NASH EQUILIBRIUM IN SEALED BID PAY-AS-BID RENEWABLE ENERGY AUCTIONS

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Abstract. Auctions have emerged globally as a crucial mechanism for acquiring renewable energy, enhancing both transparency and the expansion of clean energy initiatives. However, they differ to other auctions, and no theoretical results are available to analyze them. In this work we compute the Nash equilibrium bidding strategy for bidders participating in those auctions. By simulating the auction with bidders following this strategy and comparing the outcomes to actual results, we can gain valuable insights into market development. As an application, we analyze the German solar photo-voltaic auctions.

1. Introduction

Auctions have emerged as the dominant approach for global renewable energy procurement, accounting for nearly one-third of electricity generation in 2020 (as cited in [10]). They play a crucial role as a price discovery method contributing to the affordability and scalability of clean energy, and the continuous holding of auction rounds has resulted in observable price reductions, see [2]. According to [4], the competition in the auction is not the main factor for the price reductions, and these reductions can be attributed to both technological advancements and the maturation of the market.

Exploring the bidding behaviour of participants in renewable energy auctions has captured the interest of numerous researchers. Several models have been developed

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to analyze this behavior, see [1, 20, 16]. In [18], bidders react to their success or failure in previous rounds, and learning how to bid. Let us mention that simple agent-based models can converge to the Nash equilibrium, as proved in [3] for first- and second-price auctions.

Nash equilibrium in auctions is a well-established research area with seminal contributions from Vickrey, Milgrom, and Wilson; for a general reference, see [15]. Renewable energy auctions exhibit dynamics distinct from those observed in traditional auctions, rendering previously established results inapplicable.

In renewable energy auction the auctioneer pre-defines a target volume, denoted by T. Bidders submit bids in the form of a pair (p, v), where p represents the price per unit of energy (kWh or MWh) and v is the capacity offered. Projects are then ranked based on their unit price, with the lowest offers receiving priority. Usually, awarding continues sequentially until the target volume T is either achieved or surpassed for the first time.

Real world auctions can have hundred of participants, and show a great variability on the offered capacities, which could vary from 0.1 MW (roof top panels) to 300 MW in the same auction, as in Mexico and The Philippines [6, 7]. As a result, the number of winners can vary and it is difficult to forecast.

These is what separates the auction that we are considering here from simultaneous multi-item auctions (see [17]) where the number of winners is pre-defined; and divisible goods auctions (see [19, 21]) where participants submit prices for varying fractional shares of the item. Here the auctioneer wants to buy a volume of electricity that can be bought from one or more participants. We have described a rule for this, projects are awarded until the target volume is either achieved or surpassed for the first time, which is the typical procedure, see [1, 20]. Other rules can be used, and more complex scenarios appear in the real world since the acquired volume in a given region can be limited by transmission or grid constraints. Observe that the auctioneer can buy more capacity than the targeted one. Even if the rule from a practical point of view does not have a huge impact, it can impact the resulting equilibrium.

On the other hand, many paying rules can be used. In some countries iterative bidding formats were used as in Brazil [9], and also Vickrey-like uniform pay rules were applied, i.e., the lowest bid among the non granted projects is paid to the winners, as in Spain, among other countries, see [4, 5]. The most common format is the sealed bid, pay-as-bid auction, see [4].

Our main objective in this work is to show the computation of the Nash equilibrium strategy for bidders in the pay-as-bid scheme described, providing valuable insights into the market's level of maturity. Recently in [11] a classical iteration algorithm has been used to compute the Nash equilibrium in renewable energy auctions, but they do not prove that the algorithm converges, nor that it does to the Nash equilibrium. Our methodology integrates the resolution of differential equations with numerical simulations. This allow us to handle a general framework when modelling the population participating in renewable energy auctions.

As an application, we show how to analyze the maturity of a market by comparing the real outcomes and the Nash equilibrium. Observe that in [20] an ad-hoc rule is used as a benchmark, by considering the individual costs as the bids.

The article is organized as follows. In Section 2 we present in detail our auction model. In Section 3.1 we present some preliminaries results about the bidding strategy. In the appendix you can find the derivation of the formula. In Section 3.2 we obtain our main result, we prove that the strategy obtained is indeed the Nash equilibrium strategy. In Section 4 we apply our technique to the solar photo-voltaic (PV) auctions held in Germany beteween 2016 and 2019. Since all the information is publicly available in [8], this dataset becomes a benchmark for multiple round auctions models in renewable energy markets. The article finishes with a conclusion section where we summarize key findings and discuss their implications.

2. Model and Nash equilibrium

In this section we describe our auction model. We consider a first-price sealed-bid pay-as-bid auction. The participants submit their bids consisting of a price and a capacity to the auctioneer. The price must be below the prescribed ceiling price C.

Bids are sorted by price from the lowest to highest, ties are resolve at random. Projects are granted until the target volume T is met or exceeded for the first time. The allocation of projects follows a pay-as-bid pricing rule.

Each participant has a private volume and cost of his project. We assume the participation given by a random distribution for the number of participants and the pairs of volume and cost draw from a given distribution independently for each participant.

We assume that the distribution of the costs has no atoms. And that they are above 0 and below C since no bidder with a cost larger than the ceiling price would participate in the auction.

3. Nash equilibrium

In this section we present the bidding strategy. Then, we prove that playing according to it gives a Nash equilibrium. Finally, we make some comments related to how to compute it in a concrete example.

3.1. The bidding function B.

We begin by introducing our bidding strategy candidate formula. We have included the computation leading to this candidate formula in section 6.1 in the Appendix. In this section we will prove some lemmas leading later to the proof of Theorem 3.1 where we conclude that indeed playing according to the given formula gives a Nash equilibrium.

Given the distributions for the number of participants and the pairs of sizes and costs, we get a probability $\mathbb{P}$ over the space of sets of bidders $\{(c_i, v_i)\}_{i=1,\ldots,n}$. Here n is the number of bidders, c_i their cost and v_i their volume. If T is the target volume, we define

$$G(c) = \mathbb{P}\left(\sum_{\substack{i \leq n-1 \\ c_i < c}} v_i < T \right)$$

and we consider the bidding strategy given by

$$B(c) = c + \frac{1}{G(c)} \int_c^C G(x)\, dx \tag{1}$$

where c denotes the cost. That is, the player bids $b = B(c)$ where c is the cost of production. Observe that a priori the bid could depend on the player volume but it turn out that the strategy in our result only depends in the cost. Observe that if the bidding strategy is strictly increasing, the probability of a project been acquired is given by G. We show that although B is not necessarily strictly increasing it turns out that G gives us the probability of a project been acquired.

Let us note that B is not defined if $G(c) = 0$. We consider

$$\overline{c} = \inf\{c_0 : \mathbb{P}(c \leq c_0) = 1\}.$$

That is the smallest number such that with probability 1 every other project has a smaller cost.

A project with a cost $c < \overline{c}$ has a positive probability that every other project has a larger cost and therefore $G(c) > 0$. Also, there are projects with cost $c \geq \overline{c}$ with probability 0, so we have $G(c) = G(\overline{c})$ for $c \geq \overline{c}$. If $G(\overline{c}) > 0$, by (1), we get $B(c) = C$ for every $c \geq \overline{c}$. Motivated by that, we define $B(c) = C$ for every $c \geq \overline{c}$. We have

$$B(c) = \begin{cases} c + \frac{1}{G(c)} \int_c^C G(x)\,dx & \text{if } G(c) > 0 \\ C & \text{if } G(c) = 0 \end{cases} \tag{2}$$

In both cases $B(c) = C$ for every $c \geq \overline{c}$. Now we have a complete candidate formula for the bidding strategy.

We can observe that $B(c) \geq c$ as expected. Similarly since G is decreasing we can show that $B(c) \leq C$. Also if the participation is too low and every bid will be accepted that is $G \equiv 1$, we get

$$B(c) = c + \frac{1}{1}\int_c^C 1\,dx = c + (C - c) = C.$$

We begin with some toy examples to illustrate the resulting bidding functions.

Example 3.1. We consider an scenario with N bidders competing in a 100 MW auction with a ceiling price of 80. Each of them has a private cost distributed uniformly in $[30, 40]$ and project size distributed uniformly in $[10, 20]$. We consider $N = 7, 9, 11$.

The functions in Figure 1 and Figure 2 are continuous. However, the function in Figure 3 is not continuous since $B(40) = 80$ (we are not drawing that point there). Observe that in Figure 2 the function is continuous but has a big slope close to 40.

The key point here is whether a project with cost $\overline{c} = 40$ has a positive probability of being acquired or not. With $N = 11$ a project with a cost of 40 has probability 0 of being acquire. This is because the other projects will have a lower cost (and therefore a lower bid since B is strictly increasing) with probability 1 and since their size is at least 10, being 10 projects they will cover the 100 MW target. In the other cases we estimate the probability by 0.92 for $N = 7$ and 0.0062 for $N = 9$.

Recall that $G(c) > 0$ for every $c < \overline{c}$ and that the distribution of c does not have atoms, then the function G is continuous and therefore B is continuous.

We have $B(\overline{c}) = C$. In the case that $G(\overline{c}) > 0$ (there is a positive probability that every project is acquire) we get that B is continuous. If $G(\overline{c}) = 0$, then $\lim_{c \to \overline{c}^-} B(c) = \overline{c}$ and the function is not continuous if $\overline{c} \neq C = B(\overline{c})$. Observe that in this case the value of $B(\overline{c})$ should be above $\overline{c}$ and therefore is irrelevant since a project with such a bid has a probability of 0 of being acquire.

Lemma 3.1. *The function B given by (2) is non decreasing.*

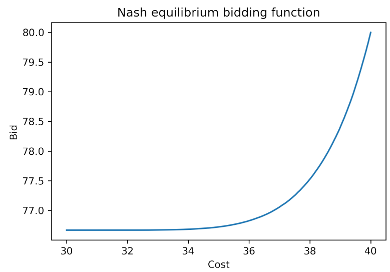

FIGURE 1. $N = 7$

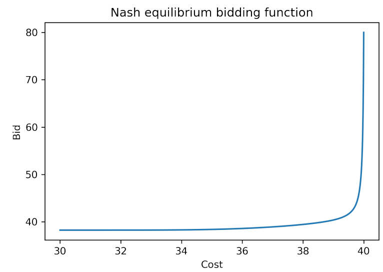

FIGURE 2. $N = 9$

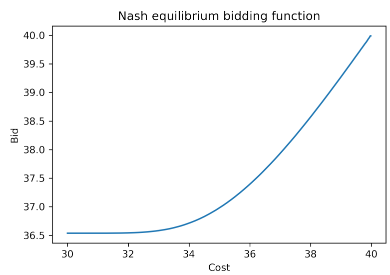

FIGURE 3. $N = 11$

Proof. If $y \geq z$ we have

$$\begin{aligned} y + \frac{1}{G(y)} \int_y^C G(x)\,dx \geq & z + \frac{1}{G(z)} \int_z^C G(x)\,dx \\ y - z \geq & \left(\frac{1}{G(z)} - \frac{1}{G(y)} \right) \int_y^C G(x)\,dx + \int_z^y \frac{G(x)}{G(z)}\,dx. \end{aligned}$$

Since $G(y) \leq G(z)$, the first term in the right hand side is non positive. Also as G is decreasing we can bound $1 \geq \frac{G(x)}{G(z)}$ in the second term proving the desired inequality. $\square$

The function B is not necessarily increasing. To illustrate such a possibility we consider the following example.

Example 3.2. We consider an scenario with 10 bidders competing in a 100 MW auction with a ceiling price of 80. The pairs of costs and sizes are distributed as follows

- With probability $1/4$ the size is 1 and the cost is uniformly distributed in $[10, 20]$
- With probability $1/4$ the size is 20 and the cost is uniformly distributed in $[20, 30]$
- With probability $1/4$ the size is 1 and the cost is uniformly distributed in $[30, 40]$
- With probability $1/4$ the size is 20 and the cost is uniformly distributed in $[40, 50]$

The function B is given in Figure 4.

For $c \leq 20$ we have $G(c) = 1$ since the projects with smaller cost have size 1 and therefore can not complete the target. Observed that if the projects with cost between 20 and 30 have not reached the target, the ones with cost between 30 and 40 can not complete it, so we have $G(c) = G(30)$ for every $c \in [30, 40]$.

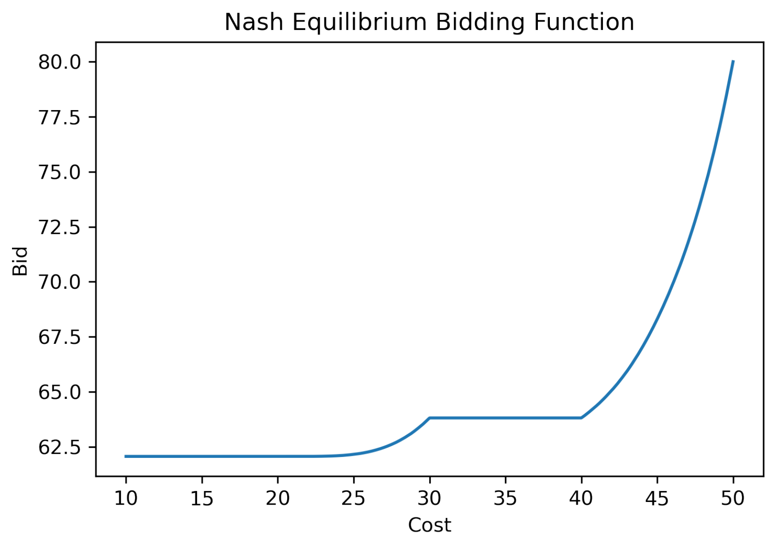

FIGURE 4. Nash equilibrium bidding function for Example 3.2.

Since B can be locally constant, ties can occur with positive probability. In that regard we state the following result for costs with the same value for B.

Lemma 3.2. *We have $B(y) = B(z)$ if and only if $G(y) = G(z)$.*

Proof. In the case that $G(y), G(z) > 0$ the result follows with an analogous computation to the one in Lemma 3.1. It remains for us to show that if $G(y) > 0 = G(z)$ then $B(y) \neq C = B(z)$.

Since $G(z) = 0$ and G is continuous we conclude that

$$\int_y^C G(x)\,dx < (C - y)G(y).$$

So, we obtain

$$B(y) = y + \frac{1}{G(y)} \int_y^C G(x)\,dx < C.$$

□

This tell us that when bidding $B(c)$, a possible tie with a project with a cost $\tilde{c}$, with $B(\tilde{c}) = B(c)$, does not change the probability of the project been acquired. In fact if $y = \min\{x : B(x) = B(c)\}$ and $z = \max\{x : B(x) = B(c)\}$ we have that the probability of of the project been acquired when bidding $B(c)$ is between $G(y)$ and $G(z)$ but those numbers coincide. Furthermore, we conclude that changing the tie-breaking rule does not alter the result.

Corollary 3.1. *In the set-up presented in Section 2 if every player bids according to $B(c)$ we have that each project is acquired with probability $G(c)$.*

3.2. **Nash equilibrium.** Our goal here is to prove that the strategy of playing $B(c)$ gives a Nash equilibrium. To derive the formula for B we assumed B to be strictly increasing and differentiable; however, we do not make such assumptions here.

Theorem 3.1. *The strategy B given by* (2) *is a Nash equilibrium in the set-up presented in Section 2.*

Proof. To prove that B is a Nash equilibrium we have to prove that the expected earnings are maximized by bidding $B(c)$.

We observe that we can assume that $b = B(\tilde{c})$ for some $\tilde{c}$. If $b < B(0)$, the project has probability 1 of being acquire but that is also the case for the bid $B(0)$. So we can assume that that does not happen, that is $b \geq B(0)$. So if B is continuous we are done.

It remains the case where B is not continuous, that is when $G(\overline{c}) = 0$. Then we can see that $\lim_{c\to\overline{c}^-} B(c) = \overline{c}$. If we bid above $\overline{c}$ the probability of winning is 0 and therefore our expected earnings are 0. We can assume that that is not the case and therefore that $b = B(\tilde{c})$ for some $\tilde{c}$.

So we have to prove that

$$(B(\tilde{c}) - c)G(\tilde{c}) \leq (B(c) - c)G(c)$$

for every $\tilde{c}$. Recalling the definition of B given by (2), we get

$$(C - c)G(\tilde{c}) + \int_{\tilde{c}}^{C} G(x)\,dx \leq \int_{c}^{C} G(x)\,dx$$

If $\tilde{c} > c$ we get

$$(\tilde{c} - c)G(\tilde{c}) \leq \int_{c}^{\tilde{c}} G(x)\,dx$$

which holds since G is decreasing. The case $\tilde{c} < c$ follows analogously. □

3.3. **Computation of G.** Depending on the distribution that the population of bidders follows, to explicitly compute G can be challenging. Let us begging with a simple example.

Example 3.3. Suppose n participants are competing for k contracts (we can think that a volume of k is been auctioned and each participant has a capacity of 1). Assume the cost of production for the participants is given by a distribution with cumulative distribution function F. To be able to win a bidder must have i with $0 \leq i \leq k-1$ competitors with better price. The probability of having exactly i of the other $n-1$ with a lower price is $\binom{n-1}{i}F(c)^i(1-F(c))^{n-1-i}$. Therefore we get

$$G(c) = \sum_{i=0}^{k-1} \binom{n-1}{i} F(c)^i (1-F(c))^{n-1-i}.$$

In the particular case that $k = 1$ and the cost is uniformly distributed between 0 and 1 we get $G(c) = (1-c)^{n-1}$. So

$$B(c) = c + \frac{1}{(1-c)^{n-1}} \int_{c}^{1} (1-c)^{n-1}\,dx = c + \frac{1-c}{n}.$$

In general, even if it is possible to do the computation explicitly this can be computational expensive. To overcome this difficulty we compute G numerically performing multiple simulations.

In each simulation, we select n following the given distribution and the sizes and costs for $n-1$ participants. We grant projects as if the bidders were bidding their costs, we record the value of the highest accepted bid (we record C if the target was not reached). From the point of view of the *bidder that we removed*, their project would need to have a cost lower than the recorded highest accepted bid to be acquired.

We perform multiple simulations and keep the values of the highest accepted bid in each of them. Then for a given cost we can estimate $G(c)$ as the number of times the last accepted bid was higher than c, that is, the project with cost c would have been acquire. Once we obtained G numerically we can compute B employing (2).

4. Germany

We analyze the solar PV auctions held in Germany. The auction results and participation can be found in [8]. The 2015 auction where conducting a uniform pay rule, later a pay-as-bid rule was implemented. We consider the rounds held between 04/2016 and 02/2019.

To employ our technique we have to provide a model for the participating population. We estimate the cost of the participant with the values from [12, 14, 13], interpolating geometrically between them. Based on the sizes of the participants we model the population as a 3% raging from 0.1 to 0.75 MW, 23% from 0.75 to 2 MW, 33% from 2 to 5 MW, 38% from 5 to 10 MW and 3% from 10 to 40 MW. The main ingredient to account for is the over all submitted volume. Therefore, to determine the number of participants, we divide the volume submitted in the round by 5.084, which is the average volume in the population.

As outlined in our methodology, we conduct multiple simulations (10000) to calculate G, and an additional set of simulations (10) for the auction, to account for the stochastic nature of our model. We present the comparison between the average obtained price and the average awarded one in the auctions in Figure 4.

In the first round in 2016 the mean awarded price was 7.41 ct/kWh, whereas our model estimated a price of 5.77 ct/kWh. As the auction rounds progress, we observe the mean awarded price moving closer towards the estimated price predicted by the model. In the last round, our model estimated a price of 4.64 ct/kWh, while the actual mean awarded price stood at 4.8 ct/kWh.

Remark 4.1. The main ingredient determining the obtained price is the competition level. That is, the quotient of the offered volume divided by the auctioned volume.

For example, we consider an auction with a 100 MW target a cost raining from 40 to 50 and a ceiling of 80. If there is only one participant then he will bid the ceiling price 80. But if we have more than one the impact of the exact population in the price is smaller as long as the competition level is maintained. With 2 participants of size 100 each we get 47.05, with 20 participants of size 10 each we get 45.03 and with 200 participants of size 1 each we get 45.01.

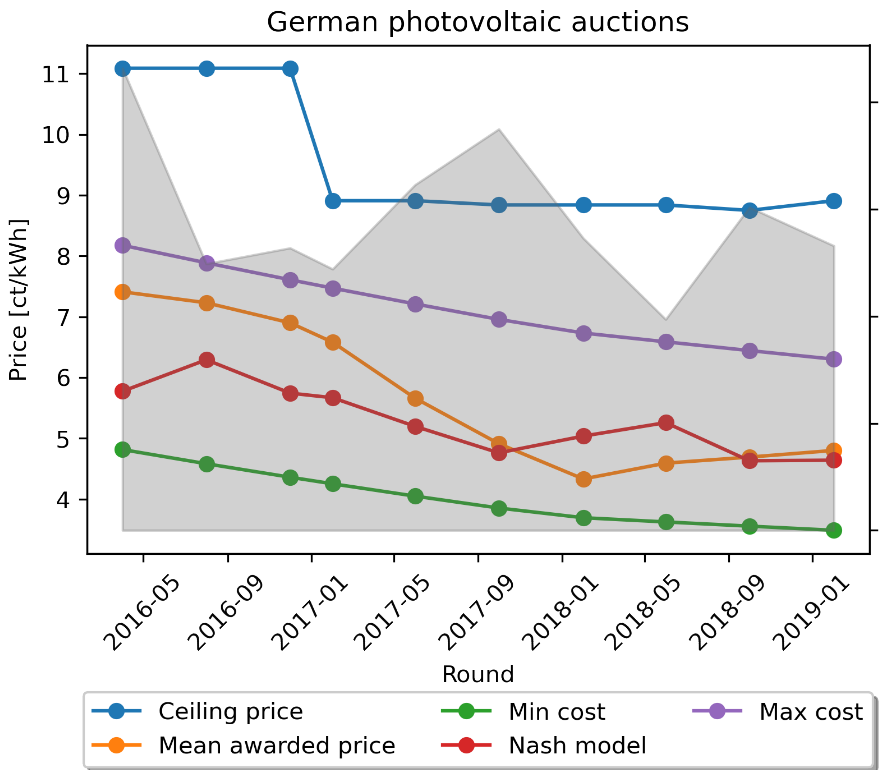

FIGURE 5. Comparison of the mean awarded price and the price obtained with the Nash model.

5. Conclusions

In this paper we compute the Nash equilibrium in sealed bid, pay-as-bid, renewable auctions. Let us recall that bidders submit a capacity, and the unitary price of each unit of energy. The projects are listed in increasing order of price, and each project is awarded if the total capacity of all the previous ones has not met the target volume.

Our technique allow us to compute the strategy corresponding to a Nash equilibrium. We illustrate the shape of the bidding function B in an example in Figure 1. This function resembles the one presented in [11], and we can conjecture that their algorithm would converge to the function we have obtained. However, since the uniqueness of the Nash equilibrium is unknown, we can not assert this with certainty. Also, algorithmic collusion is possible, and the iterative methods could converge to a different strategy.

With the bidding function we can study an auction to compare the price obtained with the expected revenue given by the Nash equilibrium. We analysed the solar PV auction conducted in Germany. As the auction rounds progress, we observe the mean awarded price gradually approaching towards the estimated price predicted by the Nash equilibrium of the model.

The difference between the price predicted by the model and the actual price offers valuable insights into the development level of the market. This understanding of the speculative component of the price can aid decision-makers in making informed and strategic choices. For instance, a reduction in the difference between the obtained price and the model's prediction can serve as an indicator, signalling that the market may be ripe for an increase in the auctioned volume.

6. Appendix

Here we include the computation that derives the formula for the Nash equilibrium. Then, we also include the multiple round case.

6.1. **Derivation of the formula.** Here our goal is to derive the bidding strategy given by the symmetric Nash equilibrium. Observe that the assumptions that we make here are only used to derive the formula but are not part of the hypothesis of our main result Theorem 3.1 where we prove that the derived formula gives indeed the Nash equilibrium.

Let us remark that the outcome of the auction for a participant bidding b (his project been acquired or not) remains independent of the volume of that specific project. Motivated by that, we search for a bidding strategy for the players given as a function B of the cost of production. That is, the player bids $b = B(c)$ where c is the cost of production.

We assume that B is strictly increasing so that the lower cost bidders will have the lower bids, and also the probability for a tie is 0. Therefore the probability of successfully biding only depends on the cost. We denote G the function of the cost that gives the probability of the project being acquired.

Given the distributions for the number of participants and the pairs of sizes and costs, we get a probability $\mathbb{P}$ over the space of sets of bidders $\{(c_i, v_i)\}_{i=1,\ldots,n}$. Here n is the number of bidders, c_i their cost and v_i their volume. If T is the target volume, we have

$$G(c) = \mathbb{P}\left(\sum_{\substack{i\leq n-1\\ c_i<c}} v_i < T\right).$$

The cumulative distribution of the bids is given by $G(B^{-1}(\cdot))$. So the expected utility when biding b is given by

$$(b-c)G(B^{-1}(b)).$$

As we mentioned our goal is to derive a formula for B, to that end, we differentiate the expected utility with respect to b, we get

$$G(B^{-1}(b)) + \frac{(b-c)G'(B^{-1}(b))}{B'(B^{-1}(b))}.$$

If B is given by a Nash equilibrium, for $b = B(c)$ this expression must be zero. We get

$$\begin{aligned} G(c) + \frac{(B(c)-c)G'(c)}{B'(c)} &= 0 \\ B'(c)G(c) + (B(c)-c)G'(c) &= 0 \\ B'(c)G(c) + B(c)G'(c) &= cG'(c) \\ (B(c)G(c))' &= cG'(c) \end{aligned}$$

Integrating between c and C we get

$$B(C)G(C) - B(c)G(c) = \int_c^C xG'(x)\,dx.$$

Integrating by parts we get

$$B(C)G(C) - B(c)G(c) = CG(C) - cG(c) - \int_c^C G(x)\,dx.$$

We can assume that $B(C) = C$, that is, bidders with cost C bid the ceiling price. We get

$$B(c) = c + \frac{1}{G(c)} \int_c^C G(x)\,dx \tag{3}$$

We have obtained a candidate formula for the Nash equilibrium.

6.2. **Multiple rounds case.** Here we tackle the general N rounds case. Here the bidders are allow to participate in the next round if their bid has been rejected. Therefore, their behaviour in a given round is influence by the existence of future rounds.

Computing the Nash equilibrium strategy for a given round requires knowledge about the participation in future rounds. Due to this, we find these computations less suitable for our context, where different auction rounds are separated by several months or even years, making future predictions highly uncertain.

In section 4 we have analyzed a multi round auction. Given the previously discussed considerations, we computed the Nash equilibrium for each round independently. Nevertheless we belive that the computation that we perform here might be relevant in a different context.

We consider $G_1, \dots, G_N$, the functions that give the probability of successfully biding with a cost c for each round, and we want to calculate $B_1, \dots, B_N$ alongside $U_1, \dots, U_N$. Here B_k denotes the Nash equilibrium bidding strategy as a function of the cost when bidding in round k. Also U_k denotes the expected utilities if a bidder stars participating in round k, that is the earnings can be the product of their project been acquire in the rounds from k to N.

Because of the computation for one round, we have

$$B_N(c) = c + \frac{1}{G_N(c)} \int_c^C G_N(x)\,dx,$$

$$U_N(c) = \int_c^C G_N(x)\,dx$$

and $U_N'(c) = -G_N(c)$.

Now we move to the general k-th round for $k < N$. When bidding b, the expected utility is given by

$$\begin{aligned}(b-c)G_k(B_k^{-1}(b)) &+ (1-G_k(B_1^{-1}(b)))U_{k+1}(c) \\ &= (b-c-U_{k+1}(c))G_k(B_k^{-1}(b)) + U_{k+1}(c)\end{aligned}$$

By differentiating this quantity we get

$$G_k(B_k^{-1}(b)) + \frac{(b-c-U_{k+1}(c))G_k'(B_k^{-1}(b))}{B_k'(B_k^{-1}(b))}$$

If B_k is given by a Nash equilibrium, for $b = B_k(c)$ this expression must be zero, so we get

$$\begin{aligned}G_k(c) + \frac{(B_k(c)-c-U_{k+1}(c))G_k'(c)}{B_k'(c)} &= 0 \\ G_k(c)B_k'(c) + (B_k(c)-c-U_{k+1}(c))G_k'(c) &= 0 \\ G_k(c)B_k'(c) + B_k(c)G_k'(c) &= (c+U_{k+1}(c))G_k'(c) \\ (G_k(c)B_k(c))' &= (c+U_{k+1}(c))G_k'(c)\end{aligned}$$

By integrating we get

$$\begin{aligned}-G_k(c)B_k(c) &= \int_c^C (x+U_{k+1}(x))G_k'(x)\,dx \\ -G_k(c)B_k(c) &= -(c+U_{k+1}(c))G_k(c) - \int_c^C (1+U_{k+1}'(x))G_k(x)\,dx\end{aligned}$$

So

$$B_k(c) = c + U_{k+1}(c) + \frac{1}{G_k(c)}\int_c^C (1+U_{k+1}'(x))G_k(x)\,dx$$

and

$$U_k(c) = \int_c^C (1+U_{k+1}'(x))G_k(x)\,dx + U_{k+1}(c).$$

Differentiating we get

$$\begin{aligned}U_k'(c) &= -(1+U_{k+1}'(x))G_k(x) + U_{k+1}'(c) \\ &= -G_k(x) + (1-G_k(c))U_{k+1}'(c)\end{aligned}$$

And with this formula we can obtain by reverse induction that

$$1 + U_k' = \prod_{i=k}^{N} 1 - G_i.$$

For $k = N$ we get $1 + U'_N = 1 - G_N$ which holds. And if we assume the hypothesis for $k + 1$, for k we get

$$\begin{aligned}1 + U'_k &= 1 - G_k + (1 - G_k)U'_{k+1} \\ &= 1 - G_k + (1 - G_k)\left(-1 + \prod_{i=k+1}^{N} 1 - G_i\right) \\ &= (1 - G_k)\prod_{i=k+1}^{N} 1 - G_i \\ &= \prod_{i=k}^{N} 1 - G_i\end{aligned}$$

so the result follows. Therefore we can write

$$B_k(c) = c + U_{k+1}(c) + \frac{1}{G_k(c)}\int_c^C \left(\prod_{i=k+1}^{N} 1 - G_i(x)\right) G_k(x)\, dx$$

and

$$U_k(c) = \int_c^C \left(\prod_{i=k+1}^{N} 1 - G_i(x)\right) G_k(x)\, dx + U_{k+1}(c).$$

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