Mathematical Models and Methods in Applied Sciences

Kinetic theory of active particles meets auction theory

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In this paper we study Nash equilibria in auctions from the kinetic theory of active particles point of view. We propose a simple learning rule for agents to update their bidding strategies based on their previous successes and failures, in first-price auctions with two bidders. Then, we formally derive the corresponding kinetic equations which describe the evolution over time of the distribution of agents on the bidding strategies. We show that the stationary solution of the equation correspond to the symmetric Nash equilibrium of the auction, and we prove the convergence to this stationary solution when time goes to infinity. We also introduce a more general learning rule that only depends on the income of agents, and we apply to both first- and second-price auctions. We show that agents learn the Nash equilibrium in first-price and second-price auctions with these rules. We present agent based simulations of the models, and we discuss several open problems.

Keywords: kinetic theory; active particles; game theory; auctions; collective learning; evolutionary economics.

AMS Subject Classification: 91A22, 91A26, 91B26

1. Introduction

The kinetic theory of active particles (KTAP) is a relatively new field of research, having been developed in the last years. It provides a theoretical framework for modeling social and economic systems, based on the tools and ideas of statistical mechanics.[2,5,7] The key idea behind the kinetic theory of active particles is to model any system as a collection of interacting agents, each having its own internal state and dynamics. The evolution of the system is then described by a set of integro-differential equations of Boltzmann type,[27] which take into account the interactions between the agents as well as their internal dynamics, and the interactions of the agents with the environment. Hence, we can explore very large spaces of strategies starting with simple microscopic rules. In the grazing limit, the Boltzmann-type equations can be simplified to a Fokker-Planck equation that describes the evolution of the probability distribution function of the agents' internal states. The KTAP is a very powerful tool for modeling a wide variety of phenomena, including opinion formation,[29,30,40] finance and economics,[21,28] traffic flow,[13,43,44] swarm and flocking behavior,[4,6,12], learning[9,34] life science problems,[1,3] and the spread of epidemics[14,22], among many others[15,17,26,39].

On the other hand, game theory is a branch of mathematics that studies the optimal behavior of rational agents in strategic situations.[16] Strategic situations involve agents whose actions are interdependent, shaped not only by each other but also by unpredictable factors from the environment, the so-called Nature's moves. The rationality hypothesis in game theory assumes that agents are able to perfectly analyze the strategic situation and compute the optimal way to play. When they reach a situation where nobody can improve by unilaterally changing their actions, we say that they are playing a Nash equilibrium, see Section §2 for more details on terminology and relevant results.

However, in real world games, agents are often not perfectly rational. They may have limited cognitive abilities, incomplete information about the strategic situation, or may be influenced by emotions or biases. As a result, agents in real-world games often learn to play by trial and error, educated guesses, and heuristic plans.[18,37,38,41]

Agents typically try to learn optimal strategies by improving their play based on the success or failure of their past actions, a process known as reinforcement learning. Over time, agents learn to associate certain actions with higher rewards and lower penalties, and they begin to select those actions more frequently. Other dynamics that can be used include imitation learning, evolutionary processes, and best-response dynamics, especially for large spaces of strategies.[8,10,19] Let us stress that the convergence to Nash equilibria cannot be guaranteed; for instance, all known dynamics fail for the simple and well known Tic-Tac-Toe game. The kinetic theory associated to the evolution of learning agents was used recently mainly for zero-sum games.[31,32]

Game theory plays a significant role in the kinetic theory of active particles

(KTAP).[1] Interactions between particles are modeled using game-theoretic tools, often resulting in non-local, non-linear dynamics. Furthermore, the strategies employed by particles, driven by their internal activity, are likely to be heterogeneously distributed across the population, as well as the corresponding payoffs. Importantly, this distribution of strategies is not static; it evolves over time due to the learning capabilities of the agents. These features are crucial to KTAP's ability to effectively model complex systems.

Several interesting problems in game theory are related to auctions.[20,23,24,35,36] In a typical auction setting, bidders have private valuations of the goods being auctioned. These valuations are usually unknown to other bidders and may follow some unknown probability distribution. Additionally, the number of bidders may be uncertain.

Under certain assumptions on the private valuations of the goods, it is possible to compute the optimal bidding strategy for a given bidder. This strategy typically involves bidding a certain percentage of the bidder's own valuation, that depends on the specific auction format. For example, in a second-price auction, the optimal bidding strategy for a bidder is to bid their true valuation. This is because the bidder will win the auction if their bid is higher than all other bids, and they will only pay the second-highest bid. However, in a first-price auction, the bidder will pay their own bid if they win, and the optimal bidding strategy in a first-price auction is to bid a percentage of the bidder's true valuation that is less than 100% and depends on the number of bidders, see Section §2 for details.

Hence, in Section §3 we propose and analyze a simple toy model of learning for two players in a first-price auction, from the KTAP point of view. We choose a simple update rule for each player's bidding strategy, depending on the success or failure of their previous bid, and the strategies they use. The corresponding kinetic equations are then obtained and we show that the Nash equilibrium of the auction is a stationary solution. We provide an heuristic argument of the long-time asymptotic behavior of the model, which coincides with agent based simulations showing that bidders learn to play the Nash equilibrium.

We analyze a more realistic model in Section §4. We consider both first-price and second-price auctions with a learning rule that depends only on the profit that bidders make in the auction. We do not assume, as before, that the bidders know the other player strategy, which is a strong assumption since only bids are public information, and they include both the strategy and the valuation of the players. We show the convergence to the corresponding Nash equilibria in both cases.

Let us observe that this modeling framework not only captures the essence of KTAP, but also bridges the gap between KTAP and game theory, providing valuable insights into the dynamics of complex systems where learning and strategic interactions are at play. We close the paper by discussing some generalizations and open problems in Section §5, showing that the model can be extended to k-players auctions, and discussing several lines to explore, aiming to extend both KTAP and

Game Theory.

2. Preliminary definitions

2.1. *Game theoretic notions*

Let us recall briefly the definition of a game. For details, we recommend the books of Krishna and Maschler[23,24].

Definition 2.1. [23] A *game* G consists in:

(1) a set of players $\mathcal{N}$;
(2) for each player $i \in \mathcal{N}$ there exists a set of *actions* $\mathcal{A}_i$, or *pure strategies*;
(3) for each player $i \in \mathcal{N}$ there exists a function $u_i : \prod_{j \in \mathcal{N}} \mathcal{A}_j \to \mathbb{R}$, called the utility function.

As usual in Game Theory, given a vector x, we call x_{-i} the vector x without the i-th component. We will identify x and (x_i, x_{-i}).

Definition 2.2 (Nash Equilibrium). A *Nash equilibrium* in a game G is a vector $a^* \in \prod_{j \in \mathcal{N}} \mathcal{A}_j$ such that

$$u_i(a^*) \geq u_i(a_i, a^*_{-i}).$$

for all player i, and any $a_i \in \mathcal{A}_i$.

Essentially, the players have no incentive to modify unilaterally their actions. This definition can be extended to mixed strategies, where instead of an action, each player i selects a probability distribution on the actions $\mathcal{A}_i$, and the utility function is defined in terms of the expected value of the utility given the players' mixed strategies.

2.2. *Auction theory*

In single-object auctions, a set of bidders compete to acquire a single object. Each bidder has a private valuation for the good, which is the value that the object represents to them. Private valuations mean that each bidder does not know the valuations of the other bidders.

We also assume that the valuations are randomly generated from a known set of independent and identically distributed random variables.

Formally, a single-object auction can be modeled as follows:

(1) Let $\mathcal{N}$ be the set of bidders.
(2) Let $\mathbb{V}$ be the known random variable with continuous distribution f, from which the valuations are generated.
(3) Let $v_i \geq 0$ be bidder i's private valuation for the auctioned good.
(4) Let b_i the bidder i's bid.

We then define a game G by introducing for each bidder i (i) a set of action A_i, (ii) the amount b_i that player i can bid modeled by a function $v_i \in \mathbb{R} \mapsto \beta(v_i) = b_i \in \mathbb{R}$, and (iii) the utility functions defined as the sum a loosing bidder must pay, and, if i won, the difference between the private value v_i and the amount the winner has to pay.

We need to define a mechanism to decide which bidder is the winner. There are several auctions rules, but we will concentrate only on the most common ones.

A *first-price auction* is a type of auction in which the player with the highest bid wins the auctioned object and pays his bid, and the loosing bidders do not pay anything. First-price auctions can be conducted either as sealed-bid auctions or as an iterative process like English auctions. In a sealed-bid first-price auction, all bidders submit their bids simultaneously in sealed envelopes, while in an English first-price auction, the auction starts at a low price and bids are called out orally. The auction ends when no player is willing to bid any higher. In this type of auctions, the Nash equilibrium is known, and depends on the number of bidders. For instance, given N bidders with independent and identical valuations $\{v_i\}_{1\le i\le N}$ with uniform distribution $\mathbb{V} \sim \mathcal{U}[0,1]$, the strategy of player i in a symmetric Nash equilibrium is a function β depending on its private valuation,[23,24]

$$\beta(v_i) = \frac{(N-1)v_i}{N}.$$

In a *second-price auction*, the bidder with the highest bid wins the auction, but pays the price of the second-highest bid. This type of auction was introduced by Vickrey,[42] and the symmetric Nash equilibrium is to bid the true valuations,[23]

$$\beta(v_i) = v_i.$$

Let us remark that the hypothesis that $\mathbb{V}$ is a uniform distribution is only used for simplicity. Indeed, the Revenue Equivalence Theorem, a cornerstone in auction theory, enables us to find the equilibrium in different auctions by comparing with the second-price auction results.

Theorem 2.1 (Revenue Equivalence Theorem[23]). *In any auction where the bidder with the highest bid wins, assuming that*

(1) the valuations $\mathbb{V}_i$ are random variables, independent, and identically distributed;
(2) the bidders are risk neutral;
(3) the expected payment of bidders with valuation 0 *is* 0,

in any symmetric and increasing equilibria (i.e., β is an increasing function) the auctioneer has the same expected revenue.

We do not enter in the discussion of risk neutrality, essentially it means that bidders are not influenced by winning or not the auction, and just maximize their utilities.

3. A simple model for two players, first-price auctions

We will consider a fixed set of N bidders. In each step we choose two of them at random to participate in a sealed-bid first price auction. We assume that the probability of a fixed player i to interact in a time interval Δt follows a Poisson distribution of parameter $\lambda = 1$.

Player i's internal activity is modelled by a parameter p_i belonging to $[0,1]$ that will evolve in time as a consequence of interactions among bidders. The initial value $p_i(0)$ of bidders' activity is sampled from a random variable with a given distribution independently for every bidder.

Once that the two players are chosen, each will sample independently a valuation v_i of the object to be auctioned from random variables $\mathbb{V}_i \sim \mathcal{U}[0,1]$. However, it is important to note that this specific selection of random variables does not inherently limit the model's adaptability. It can be effectively tailored to accommodate any given probability distribution for the valuations.

Finally, each agent will bid accordingly with

$$\beta_i(v_i, t) = v_i p_i(t). \tag{3.1}$$

Thus $p_i(t)$ is the fraction of the valuation that bidder i is willing to bet. Notice that the strategy expressed by bidders are heterogeneously distributed in the population as a consequence of the individual sampling of the value.

After this, each player updates their parameters. Only players involved in the game change their parameters. Let's say i and j are the players that participated in the auction. They will update their parameters with the following rule:

$$p_j(t + \Delta t) = \begin{cases} p_j(t)q & \text{if } \beta_i(v_i, t) \le \beta_j(v_j, t) \text{ and } p_j(t) \ge p_i(t), \\ \\ p_j(t) & \text{otherwise,} \end{cases} \tag{3.2}$$

and

$$p_i(t + \Delta t) = \begin{cases} 1 - q(1 - p_i(t)) & \text{if } \beta_i(v_i, t) \le \beta_j(v_j, t) \text{ and } p_j(t) \ge p_i(t), \\ \\ p_i(t) & \text{otherwise,} \end{cases} \tag{3.3}$$

where $q \in (0,1)$ is a fixed constant and Δt is a short time period. The parameter q is a measure of how likely are the players to modify their strategies. The closer is q to 1, the less willing are the players to modify their strategies. However, if q is small we interpret that the bidder is not confident in its strategy and is willing to make huge modifications.

The motivation for this rule is the following: imagine players i and j compete, and j emerges as the winner. If j's parameter p_j is greater than or equal to i's parameter p_i, it suggests j could have potentially achieved a higher revenue by bidding less.

Since the revenue is determined by $v_j - \beta_j(v_j)$, where v_j represents their valuation and $\beta_j(v_j)$ represents their bid, a smaller parameter would have led to a lower bid, potentially increasing their revenue.

Therefore, in this scenario, it is advantageous for j to adjust their strategy and bid a smaller fraction of their valuation in future rounds. However, if j's parameter is already lower than i's, they will maintain their current strategy, as their victory might have been due to chance rather than an excessively high bid.

Conversely, in this same situation, player i, who lost despite having a lower parameter, will seek to increase their parameter in subsequent rounds to enhance their chances of winning.

Indeed we can check that the proposed rule gives an increase in the parameter as

$$1 - q(1 - p_i(t)) \geq p_i(t)$$

if and only if

$$1 - p_i(t) \geq q(1 - p_i(t)),$$

which is true since $q \in (0, 1)$.

Figure 1 shows the evolution in time of bidders' p parameter for a simulation with $N = 1000$ bidders with two different initial distribution of p parameters. The learning parameter q was set to $q = 0.995$ in both simulations. We can observe that bidders coordinate in the sense that they tend to share the same parameter, $p = 1/2$ in both cases. Varying the initial distribution of p's does not affect this qualitative behavior. These agent-based simulations thus suggest that bidders are actually learning the Nash equilibrium, namely bidding half their valuation.

To assess the influence of q, we show in Figure 2 the final distribution of bidders'p parameter for different values of q. When $q \simeq 1$, the distribution is concentrated around 0.5 in agreement with what we just observed. Then, as q decreases, the distribution is still mostly concentrated around 0.5 but its variance increased until a critical value $q \simeq 0.6$ where the distribution becomes bimodal. As q keeps on decreasing, the two peaks travel to the left and right respectively. Eventually when $q \simeq 0$, the distribution is very similar to $(1/2)\delta_0 + (1/2)\delta_1$, i.e. approximately half the bidders have $p \simeq 0$ whereas the other half have $p \simeq 1$.

The model we described embodies key features of the kinetic theory of active particles: nonlinearity, heterogeneity, and learning.

First, heterogeneity is inherent: bidders express their strategies through bids in equation (3.1) , and these bids vary based on each bidder's private valuation. This valuation, represented by the heterogeneously distributed parameter p_i, makes each bidder unique.

Second, the model exhibits nonlinearity: interactions between bidders drive the evolution of their internal activity (p_i) according to the nonlinear updating rules (3.2)-(3.3). This means each bidder continuously adapts their strategy and internal state based on the dynamic, stochastic environment created by the other bidders.

Third, learning is a core principle: through repeated interactions, bidders adjust their internal activity, effectively learning from their experiences and the behavior of others. This learning process shapes their future bidding strategies and contributes to the evolving dynamics of the system.

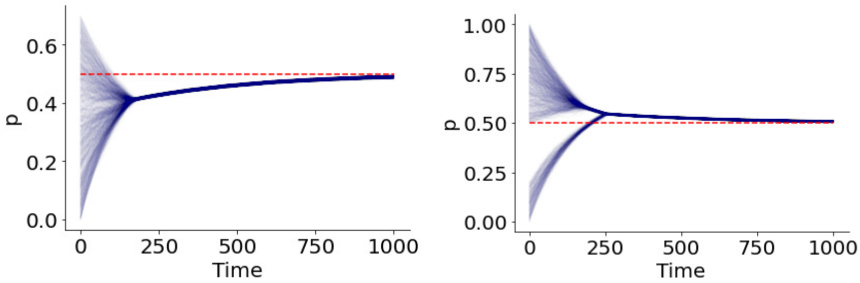

Fig. 1. Individual trajectories of bidders' p parameter, with an initial condition uniformly distributed in $[0, 0.7]$ (left), and where a quarter of the bidders draw independently their p uniformly in $[0, 0.2]$ and the remaining bidders uniformly in $[0.5, 1]$ (right).

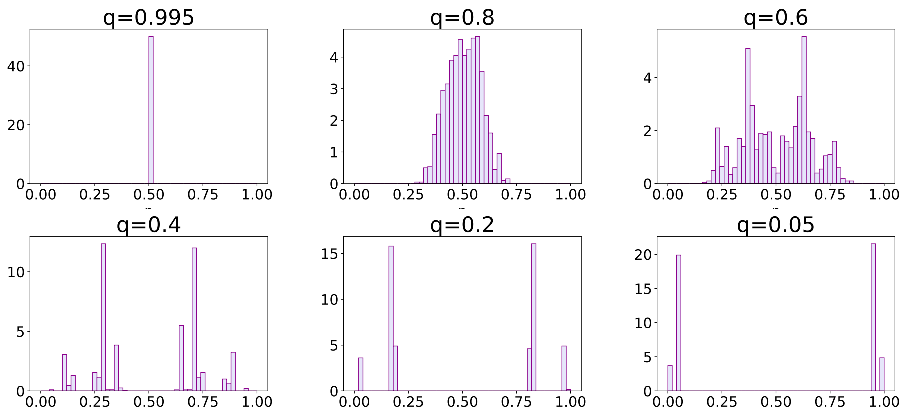

Fig. 2. Histograms of bidders' p parameter at the stationnary state for different values of q. The initial distribution of bidders' p is the same as in Figure 1 (right).

The rest of this section is devoted to a theoretical study of the model aiming at understanding why bidders actually learn to bid the Nash equilibrium.

3.1. *Evolution of the parameters*

We want to know if agents learn how to play optimally. Hence, we will be concerned with the time evolution of their parameters. To achieve this, we want to construct a transport equation for the empirical measure defined by the distribution of agents in $(0,1)$. We will do this in two steps. First, we will describe the evolution of the parameter of a fixed bidder from a time t to $t+\Delta t$, where Δt is a short time. Then, by taking $\Delta t \to 0$ we will get a differential equation and we will be able to get a first order partial differential equation (in a weak formulation) from the evolution of the empirical measure.

Let us fix a player i and write the expected value of $p_i(t+\Delta t)$:

$$p_i(t+\Delta t) = p_i(t)\mathbb{P}(i \text{ was not chosen}) + \sum_j p_i' \mathbb{P}(i \text{ plays against } j)$$

where p_i' represents the value of p_i after the interaction. The sum on the right-hand side must be split into four cases corresponding to i winning or losing, and if its parameter was bigger or smaller than the parameter of its opponent. Notice that i has j as an opponent with probability $\frac{1}{N-1}$. After rearranging the terms we get:

$$\begin{aligned} \frac{p_i(t+\Delta t) - p_i(t)}{\Delta t} = &- p_i(t) + \frac{1}{N-1}\Bigg[p_i(t) q \sum_{j:p_i \geq p_j} \mathbb{P}(p_j V_j < p_i V_i) \\ &+ (1 - q(1-p_i(t))) \sum_{j:p_j \geq p_i} \mathbb{P}(p_j V_j > p_i V_i) \\ &+ p_i(t) \sum_{j:p_i > p_j} \mathbb{P}(p_j V_j > p_i V_i) \\ &+ p_i(t) \sum_{j:p_i < p_j} \mathbb{P}(p_i V_i > p_j V_j) \Bigg]. \end{aligned} \tag{3.4}$$

The probabilities on the right hand side can be computed using the following Lemma:

Lemma 3.1. *Let $a, b \in (0,1)$ and let $\mathbb{V}_i$, $\mathbb{V}_j \sim \mathcal{U}[0,1]$ be independent random variables. Then,*

$$\mathbb{P}(a\mathbb{V}_i < b\mathbb{V}_j) = \begin{cases} b/(2a) & \text{if} \quad b < a \\ 1 - a/(2b) & \text{if} \quad b \geq a. \end{cases}$$

The proof is a direct calculation and therefore omitted.[11]

As a consequence of this lemma we obtain

$$\begin{aligned} (N-1)\frac{p_i(t+\Delta t) - p_i(t)}{\Delta t} = &\, q p_i \sum_{j:p_i \geq p_j} \left(1 - \frac{p_j}{2p_i}\right) + (1-(1-p_i)q) \sum_{j:p_j \geq p_i} \left(1 - \frac{p_i}{2p_j}\right) \\ &+ p_i \sum_{j:p_i > p_j} \frac{p_j}{2p_i} + p_i \sum_{j:p_j > p_i} \frac{p_i}{2p_j} - p_i(N-1). \end{aligned}$$

Letting $\varepsilon := 1-q$ and noticing that $1-(1-p_i)(1-\varepsilon) = p_i + \varepsilon(1-p_i)$, we can rearrange terms and get in the limit $\Delta t \to 0$ the equation

$$\frac{N-1}{\varepsilon}\frac{dp_i}{dt} = -\, p_i \sum_{j:p_i \geq p_j} \left(1 - \frac{p_j}{2p_i}\right) + (1-p_i) \sum_{j:p_j \geq p_i} \left(1 - \frac{p_i}{2p_j}\right). \tag{3.5}$$

For this limit to be valid we need to assume that q is close enough to one, so that the changes on the parameters are slow.

3.2. *Equilibrium of the discrete dynamic*

We look for a symmetric and stationary solution of (3.5):

Lemma 3.2. *The only symmetric and stationary solution to the discrete dynamic* (3.5) *is* $p_i(t) = \frac{1}{2}$ *for all i and every t.*

Proof. We replace the left hand side of (3.5) by 0 and in the right hand side we replace $p_i, p_j =: p$. As the last two sums are over sets where $p_j < p_i$ and $p_i < p_j$, equation (3.5) takes the form:

$$0 = qp \sum_{j:j\neq i} \left(1 - \frac{1}{2}\right) + (1-(1-p)q) \sum_{j:j\neq i} \left(1 - \frac{1}{2}\right) - p(N-1)$$

i.e.

$$(1-q)(N-1)\left(\frac{1}{2} - p\right) = 0.$$

The only factor that can be zero is the last one so $p = \frac{1}{2}$ is the unique solution. □

3.3. *A transport equation*

In this section we rewrite the system of equations (3.5) for the empirical measure

$$f_t = \frac{1}{N}\sum_{i=1}^{N} \delta_{p_i(t)},$$

associated to the p_i's. Recall that

$$\int \varphi(p)df_t(p) = \frac{1}{N}\sum_{i=1}^{N} \varphi(p_i(t)),$$

for any bounded measurable function φ. Then (3.5) can be written as

$$\frac{N-1}{N\varepsilon}\frac{dp_i}{dt} = H[f_t](p_i(t)),$$

where H is defined as

$$H[f_t](p) = -p\int df_t(p')\mathbf{1}_{\{p\geq p'\}}\left(1 - \frac{p'}{2p}\right) + (1-p)\int df_t(p')\mathbf{1}_{\{p'\geq p\}}\left(1 - \frac{p}{2p'}\right).$$

Via a re-scaling of the time variable, we can absorb the positive constants ε and $(N-1)/N$ to obtain

$$\frac{dp_i}{dt} = H[f_t](p_i(t)) \tag{3.6}$$

It follows that

$$\begin{aligned}\frac{d}{dt}\int \varphi(p)df_t(p) &= \frac{1}{N}\sum_{i=1}^{N}\varphi'(p_i(t))p_i'(t)\\ &= \frac{1}{N}\sum_{i=1}^{N}\varphi'(p_i(t))H[f_t](p_i(t)).\end{aligned}$$

Expressing the sum on the right hand side using f_t, we see that the system of equations (3.6) is equivalent to

$$\frac{d}{dt}\int \varphi(p)df_t(p) = \int df_t(p)\varphi'(p)H[f_t](p), \qquad \forall\, \varphi \in C^1([0,1]). \tag{3.7}$$

Notice that (3.7) is the weak formulation of the first order equation

$$\frac{\partial f_t}{\partial t} + \frac{\partial}{\partial p}\left[H[f_t](p)f_t\right] = 0.$$

This transport equation thus describes the evolution in time of the distribution of agents in the strategies p, the updating rule (3.2) being embodied in the function H defined in (3.9).

We can summarize the result obtained in this section as follows:

Theorem 3.1. *The dynamic* (3.2) *gives the transport equation:*

$$\frac{\partial f_t}{\partial t} + \frac{\partial}{\partial p}\left[H[f_t](p)f_t)\right] = 0, \tag{3.8}$$

where

$$H[f_t](p) = -p\int df_t(p')\mathbf{1}_{\{p\geq p'\}}\left(1-\frac{p'}{2p}\right) + (1-p)\int df_t(p')\mathbf{1}_{\{p'\geq p\}}\left(1-\frac{p}{2p'}\right). \tag{3.9}$$

3.4. *Stationary solutions of transport equation*

As before, we will study the stationary solutions to the equation (3.8).

We first look for stationary solution in the form of a Dirac's Delta measure δ_a, $a \in [0,1]$:

Proposition 3.1. *The Dirac's Delta measure at $\frac{1}{2}$ is the only stationary solution of equation* (3.8) *in the set* $\{\delta_a,\, a \in [0,1]\}$.

Proof. The result follows by noticing that δ_a is a stationary solution if and only if $H[\delta_a](a) = 0$. Recalling the definition (3.9) of H, we see that

$$H[\delta_a](a) = -a\left(1-\frac{a}{2a}\right) + (1-a)\left(1-\frac{a}{2a}\right) = \frac{1}{2}(1-2a).$$

Thus $H[\delta_a](a) = 0$ only when $a = \frac{1}{2}$. □

This result was expected as we know that the only symmetric Nash equilibrium for the first-price auction is to have all the players having a parameter of $1/2$ so this translates to an empirical measure $f_t = \delta_{1/2}$.

In fact we claim that $\delta_{1/2}$ is the only stationary solution to (3.8) among all probability measures over $[0,1]$. To prove this, we first find an equation satisfied by the moments $M_k(t) = \int p^k \, df_t(p)$, $k \in \mathbb{N}$, of f_t.

Proposition 3.2. *For any $k \in \mathbb{N}$ and $t > 0$, it holds that:*

$$\frac{dM_k(t)}{dt} = \iint df_t(p)df_t(\tilde{p})\mathbf{1}_{\{p\geq\tilde{p}\}}\left(1-\frac{\tilde{p}}{2p}\right)k[\tilde{p}^{k-1}-p^k-\tilde{p}^k]. \tag{3.10}$$

Proof. We take $\varphi(p) = p^k$ as a test function in equation (3.8) and we obtain

$$\begin{aligned}\frac{dM_k}{dt} =& \frac{d}{dt}\int p^k df_t(p) = \int df_t(p)H[f_t](p)kp^{k-1} \\ =& -\iint df_t(p)df_t(\tilde{p})kp^{k-1}\mathbf{1}_{\{p\geq\tilde{p}\}}p\left(1-\frac{\tilde{p}}{2p}\right) \\ &+\iint df_t(p)df_t(\tilde{p})kp^{k-1}\mathbf{1}_{\{\tilde{p}\geq p\}}(1-p)\left(1-\frac{p}{2\tilde{p}}\right).\end{aligned} \tag{3.11}$$

By exchanging p and $\tilde{p}$ in the second integral we get

$$\frac{dM_k}{dt} = \iint df_t(p)df_t(\tilde{p})\left(1-\frac{\tilde{p}}{2p}\right)\mathbf{1}_{\{p\geq\tilde{p}\}}k[\tilde{p}^{k-1}(1-\tilde{p})-p^k]$$

from which the result follows. □

It follows in particular that

$$\frac{dM_1(t)}{dt} = \iint df_t(p)df_t(\tilde{p})\mathbf{1}_{\{p\geq\tilde{p}\}}\left(1-\frac{\tilde{p}}{2p}\right)[1-p-\tilde{p}] \tag{3.12}$$

and

$$\frac{dM_2(t)}{dt} = 2\iint df_t(p)df_t(\tilde{p})\mathbf{1}_{\{p\geq\tilde{p}\}}\left(1-\frac{\tilde{p}}{2p}\right)[\tilde{p}-p^2-\tilde{p}^2]. \tag{3.13}$$

We can easily recover the result of Proposition 3.1: if we look for a stationary solution in the form of a Dirac's Delta measure δ_a then

$$0 = \frac{dM_1(t)}{dt} = \left(1-\frac{a}{2a}\right)[1-a-a] = \frac{1}{2}(1-2a)$$

from which it follows that $a = 1/2$.

We can now prove that $\delta_{1/2}$ is in fact the only stationary solution:

Theorem 3.2. *There exists a unique stationary solution f to the equation 3.8 and this is $f = \delta_{1/2}$.*

Proof. Let f be a stationary solution. Then in view of (3.12)-(3.13),

$$0 = \frac{d(M_2 - M_1)}{dt} = \iint df(p)df(\tilde{p})\mathbf{1}_{\{p \geq \tilde{p}\}} \left(1 - \frac{\tilde{p}}{2p}\right) g(\tilde{p}, p) \tag{3.14}$$

where

$$g(x,y) = 3x + y - 2x^2 - 2y^2 - 1.$$

Notice that the set $\{g \leq 0\}$ is the disk centered at $(3/4, 1/4)$ with radius $1/(2\sqrt{2})$:

$$\{g(x,y) \leq 0\} = \left\{ \left(x - \frac{3}{4}\right)^2 + \left(y - \frac{1}{4}\right)^2 \leq \frac{1}{8} \right\}.$$

It lies in $\{y \leq x\}$ and intercepts the line $y = x$ only at $(1/2, 1/2)$. It follows that $g(\tilde{p}, p) \geq 0$ in $\{p \geq \tilde{p}\}$ and is 0 only when $p = \tilde{p} = 1/2$. Thus (3.14) implies that $f \otimes f$ is supported at $(1/2, 1/2)$, i.e., $f = \delta_{1/2}$.

The proof is finished. □

3.5. *Long-time behavior of the solutions of the transport equation*

To shed light on the observed behavior of the agent-based simulations in Figure 1, we offer theoretical explanations for two key phenomena: coordination and parameter convergence.

Bidder coordination: As the simulations run, we see the support of the distribution f_t shrink to a single point. This suggests that bidders gradually coordinate their bids, clustering around a specific value. Let us recall that the updating rules (3.2)-(3.3) incentivize bidders who overbid to adjust their strategies.

Parameter convergence: We also observe a tendency for bidders' internal parameters p_i to converge towards the value $1/2$. This can be understood as the emergence of a Nash equilibrium in the bidding game. At $p_i = 1/2$, bidders experience neither an advantage nor a disadvantage, achieving a state of mutual balance. By deviating from this point, bidders risk either winning less frequently or sacrificing potential gains due to overly high bids. Therefore, repeated interactions naturally drive them towards this equilibrium point.

Let us denote by $[a(t), b(t)]$ the convex hull of the support of f_t. It follows from equations (3.8) that $b'(t) = H[f_t](b(t))$ and $a'(t) = H[f_t](a(t))$. By noticing that $1 - \frac{p}{2p'} \geq \frac{1}{2}$ if $p' \geq p$, we see that

$$H[f_t](b(t)) \leq -\frac{b(t)}{2} + \frac{1 - b(t)}{2} f_t(\{b(t)\})$$

and

$$H[f_t](a(t)) \geq -\frac{a(t)}{2} f_t(\{a(t)\}) + \frac{1 - a(t)}{2}.$$

It follows that

$$b'(t) - a'(t) \leq -\frac{1}{2}(b(t) - a(t)) - \frac{1}{2}\Big[1 - (1 - b(t)) f_t(\{b(t)\}) - a(t) f_t(\{a(t)\})\Big].$$

Noticing that $1 = f_t([a(t), b(t)]) \geq f_t(\{a(t)\}) + f_t(\{b(t)\})$ we can bound the last bracket as

$$(1 - a(t))f_t(\{a(t)\}) + b(t)f_t(\{b(t)\}) \geq 0.$$

Thus, $b'(t) - a'(t) \leq -\frac{1}{2}(b(t) - a(t))$ from which it follows that the support of f_t shrinks to a point exponentially fast: $(b(t) - a(t) \leq (b(0) - a(0))e^{-t/2}$.

In particular the distance between f_t and the Dirac's Delta mass centered at its mean value $\delta_{M_1(t)}$ is going to 0.

We can thus rewrite the equation (3.12) satisfied by $M_1(t)$ assuming that $f_t = \delta_{M_1(t)}$ up to an error term going to 0:

$$\frac{d}{dt}M_1(t) = \Big(\frac{1}{2} + o(1)\Big) - M_1(t)$$

It follows that $M_1(t) \to \delta_{\frac{1}{2}}$.

We thus have proved

Theorem 3.3. *Any solution f_t to the transport equation* (3.8) *converges to $\delta_{\frac{1}{2}}$ as $t \to +\infty$.*

3.6. *Simulations*

In this section we consider numerical simulations for the discrete dynamics with 1000 players and 1000 iterations, that is, 10^6 two-player game interactions. We used a value for the parameter $q = 0.9$, recall that as we discussed on the derivation of the transport equation, we are interested in values of the parameter q close to 1. However, we cannot expect a convergence to a pure Dirac's Delta function, due to random fluctuations and the step size. Moreover, for a small q we would obtain a bimodal distribution.

The initial conditions we will consider are:

- $\mathcal{C}_1$: $p_i(0) = 0.99$ for every $1 \leq i \leq 1000$.
- $\mathcal{C}_2$: $p_i(0) = 0.1$ for every $1 \leq i \leq 1000$.
- $\mathcal{C}_3$: $p_i(0) = 0.01$ for $1 \leq i < 5000$ and $p_i(0) = 0.99$ for $5000 \leq i \leq 1000$.

We start by comparing the means of the distributions. In Figure 3 we can see that from the initial value of the mean it evolves fast, since in less than 100 iterations we get a stable distribution. In all the cases, we can see that the mean stabilizes around 0.500 which was expected since the stationary distribution of the kinetic equations is a Dirac's Delta centered at 1/2.

In Figure 4 we get that the variance also reaches an equilibrium fast and the limiting value is close to zero. This is again consistent with the expected distribution as the Dirac's Delta has zero variance.

Finally, we can look at the histograms we get in the last iteration for the density of players in the parameter space, presented in Figure 5.

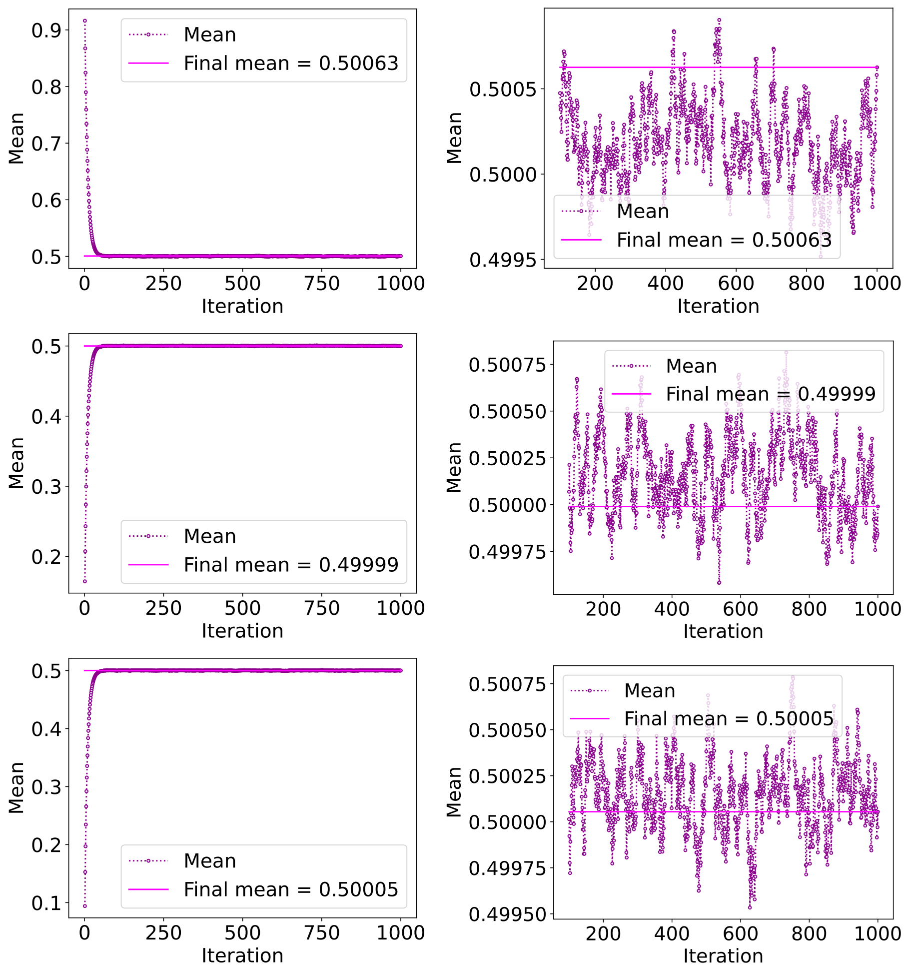

Fig. 3. We plot the evolution of the mean in 1000 iterations changing the initial condition. From left to right we have on the first row the condition $\mathcal{C}_1$, followed by a zoom of the same simulated data set after 100 iterations. The second row correspond to the initial condition $\mathcal{C}_2$. The last row corresponds to the condition $\mathcal{C}_3$.

From these results we can notice that the initial condition does not seem to be a relevant factor for the dynamics. In the three cases considered we observed that the final mean was 0.500 ± 0.001 and the final variance was 0.0015 ± 0.0005. This is the result we were expecting for long times. In all of the cases, we can notice that before 200 iterations we already achieved a distribution in those ranges for both the mean and the variance.

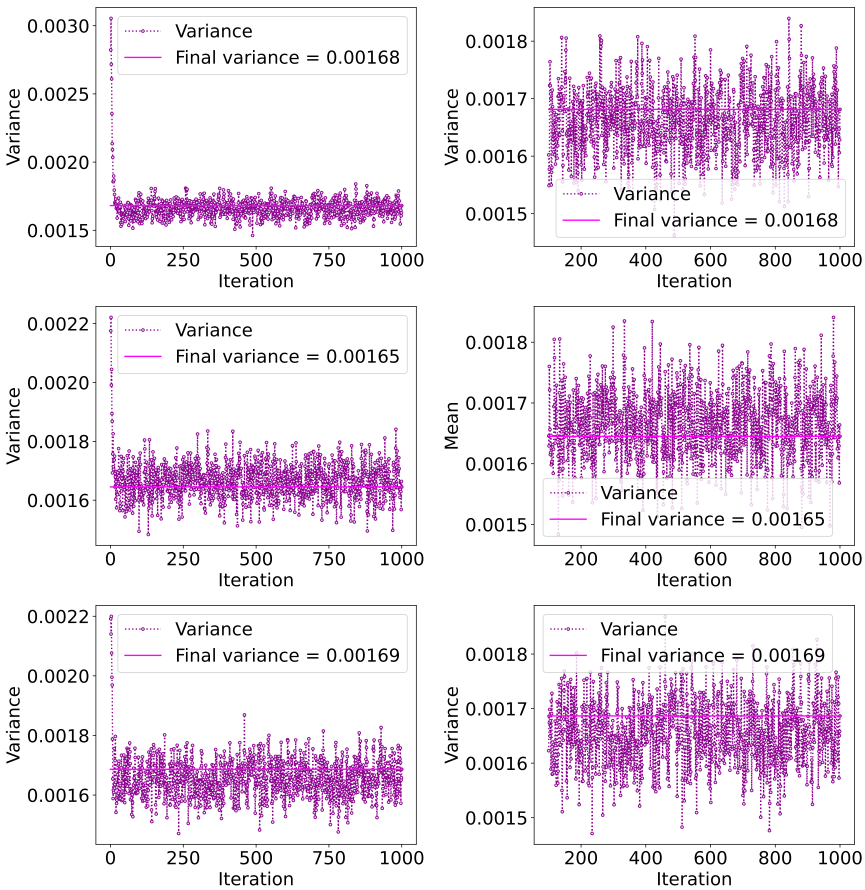

Fig. 4. We plot the evolution of the variance in the 1000 iterations changing the initial condition. From left to right we have on the first row the condition $\mathcal{C}_1$, followed by a zoom of the same simulated data set after 100 iterations. The second row correspond to the initial condition $\mathcal{C}_2$. The the last one corresponds to the condition $\mathcal{C}_3$.

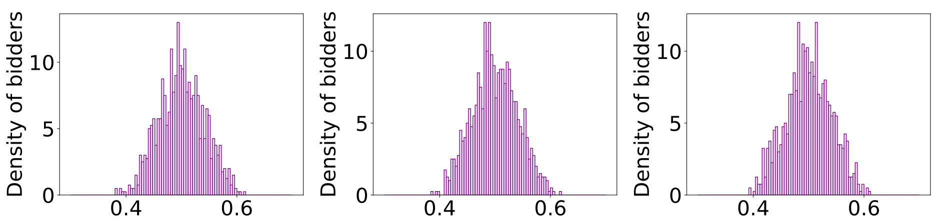

Fig. 5. Histograms of the final density of players in the parameter space, corresponding to the initial condition $\mathcal{C}_1$ (left), $\mathcal{C}_2$ (center) and $\mathcal{C}_3$ (right).

4. A more general model

Let us consider again a population of N players. As before, interactions occur following a Poisson process of rate equal to 1, and two players i and j are randomly paired in a two-player auction. They draw their valuations v_i, v_j of the object to be sold from independent random variables $\mathbb{V}_i$, $\mathbb{V}_j \sim \mathcal{U}[0,1]$. and bid

$$\beta_i(v_i) = p_i(t)v_i, \qquad \text{and} \qquad \beta_j(v_j) = p_j(t)v_j.$$

where $p_i(t), p_j(t) \in [0,1]$. The winner of the auction is the player with the highest bid. Players i and j then update their parameter p_i and p_j with the following rule:

$$p_i(t+\Delta t) = \begin{cases} p_i(t)(1-c\gamma) & \text{if } i \text{ wins,} \\ (1-\gamma)p_i(t)+\gamma & \text{if } i \text{ looses,} \end{cases} \tag{4.1}$$

where $\gamma > 0$ is a small parameter, the learning rate, and the constant c depends on the auction rules: $c = 1$ corresponds to a first price auction, and $c = 0$ to a second price auction.

This model is more realistic than the previous one since the players do not need to know the other players' valuations, but just react to the result of the auction and try to improve their utility. Notice that in the second price auction, the winner pays the second highest bid and so has no incentive to modify his p parameter since this would not lead to a strict increase of his utility. In contrast, in the first price auction where the winner pays his bid, decreasing his parameter p does increase his payoff. This the rationale behind the the values $c \in \{0,1\}$ in (4.1).

4.1. *A kinetic equation*

Let us derive formally a first order partial differential equation for the evolution of $f_t = \frac{1}{N}\sum_{i=1}^{N} \delta_{p_i(t)}$ in this model. We assume a player randomly interacts with a Poisson process with rate 1. Let us compute $p_i(t+\Delta t)$ for a fixed player i:

$$p_i(t+\Delta t) = (1-\Delta t)p_i(t) + p_i(t)(1-c\gamma)\mathbb{P}(i \text{ wins}) + (p_i(t)(1-\gamma)+\gamma)\mathbb{P}(i \text{ looses}).$$

Denoting A_{ij} the event that i plays against j, so that $\mathbb{P}(A_{ij}) = \Delta t/(N-1)$, we obtain

$$\mathbb{P}(i \text{ wins}) = \frac{\Delta t}{N-1}\sum_{j\neq i}\mathbb{P}(i \text{ wins}|A_{ij}) = \frac{\Delta t}{N-1}\sum_{j\neq i}\mathbb{P}(p_iV_i > p_jV_j),$$

and similarly,

$$\mathbb{P}(i \text{ looses}) = \frac{\Delta t}{N-1}\sum_{j\neq i}\mathbb{P}(p_jV_j > p_iV_i).$$

Using Lemma 3.1 to compute these probabilities leads to

$$\frac{p_i(t+\Delta t)-p_i(t)}{\Delta t} = -p_i(t)+p_i(t)(1-c\gamma)\frac{1}{N-1}\left\{\sum_{j:p_i\geq p_j}\left(1-\frac{p_j}{2p_i}\right)+\sum_{j:p_j>p_i}\frac{p_i}{2p_j}\right\}$$
$$+(p_i(t)(1-\gamma)+\gamma)\frac{1}{N-1}\left\{\sum_{j:p_j\geq p_i}\left(1-\frac{p_i}{2p_j}\right)+\sum_{j:p_i>p_j}\frac{p_j}{2p_i}\right\}.$$

If we group the sums in $p_i < p_j$, $p_j < p_i$ and $p_i = p_j$, then after reordering the expression and letting $\Delta t \to 0$, we get a system of ordinary differential equations:

$$\frac{N-1}{\gamma}\frac{dp_i(t)}{dt} = \sum_{j:p_j>p_i}\frac{1}{2p_j}\left(p_i^2(1-c)-2p_ip_j+2p_j-p_i\right)$$
$$+\sum_{j:p_i>p_j}\frac{1}{2p_i}\left(-2p_i^2c+p_ip_j(c-1)+p_j\right)+\sum_{j:p_j=p_i}\frac{1-p_i(c+1)}{2}.$$

This can be written in terms of the empirical measure $f_t = \frac{1}{N}\sum_{i=1}^N \delta_{p_i(t)}$ as

$$\frac{N-1}{N\gamma}\frac{dp_i(t)}{dt} = H_c[f_t](p_i(t)),$$

where

$$H_c[f_t](p) = \int df_t(p')\mathbb{1}_{\{p'>p\}}\frac{1}{2p'}\left(p^2(1-c)-2pp'+2p'-p\right)$$
$$+\int df_t(p')\mathbb{1}_{\{p>p'\}}\frac{1}{2p}\left(-2p^2c+pp'(c-1)+p'\right) \tag{4.2}$$
$$+\int df_t(p')\mathbb{1}_{\{p=p'\}}\frac{1-p(c+1)}{2}.$$

Then as in the previous section, we obtain that f_t is a weak solution of the transport equation

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial p}\left[H_c[f_t](p)f_t\right]=0. \tag{4.3}$$

in the sense that

$$\frac{d}{dt}\int\varphi(p)df_t(p) = \int\varphi'(p)H_c[f_t](p)df_t(p). \quad \forall\varphi\in C^1([0,1]).$$

4.2. *First price auction*

In the case of a first-price auction, we use $c = 1$ so the field H_c takes the form:

$$H_1[f_t](p) = \int df_t(p')\mathbb{1}_{\{p'>p\}}\frac{1}{2p'}\left(-2pp'+2p'-p\right)$$
$$+\int df_t(p')\mathbb{1}_{\{p>p'\}}\frac{1}{2p}\left(-2p^2+p'\right) \tag{4.4}$$
$$+\int df_t(p')\mathbb{1}_{\{p=p'\}}\frac{1-2p}{2}.$$

We can prove that the Nash equilibrium where all the players bid half of their private values is a stationary solution to equation (4.3) with H_1. Moreover, we can prove the following result:

Theorem 4.1. *The unique Dirac's Delta solution of equation* (4.3) *for* $c = 1$ *is* $f(t,p) = \delta_{1/2}(p)$.

Proof. In view of (4.3), it suffices to prove that $H_1[\delta_a](a) = 0$ if and only if $a = 1/2$. This follows from

$$H_1[\delta_a](a) = \frac{-2a+1}{2} \qquad \square$$

In fact following the same informal reasoning as before we can verify that the support of f_t shrinks to a point. Indeed (with the same notation as before),

$$\begin{aligned} a'(t) = H_1[f_t](a(t)) &= \int df_t(p')\mathbb{1}_{\{p'>a\}}\frac{1}{2p'}\left(-2ap' + 2p' - a\right) + \frac{1-2a}{2}f_t(\{a(t)\}) \\ &= \int df_t(p')\frac{1}{2p'}\left(-2ap' + 2p' - a\right) \\ &= 1 - a - \int df_t(p')\frac{a}{2p'} \end{aligned}$$

and analogously,

$$\begin{aligned} b'(t) = H_1[f_t](a(t)) &= \int df_t(p')\mathbb{1}_{\{b(t)>p'\}}\frac{1}{2b}\left(-2b^2 + p'\right) + \frac{1-2b}{2}f_t(\{b(t)\}) \\ &= \int df_t(p')\frac{1}{2b}\left(-2b^2 + p'\right) \\ &= \int df_t(p')\frac{p'}{2b} - b(t). \end{aligned}$$

Thus

$$b'(t) - a'(t) = -(b(t) - a(t)) + \int df_t(p')\frac{p'}{2b} + \int df_t(p')\frac{a}{2p'} - 1.$$

Since both integrals on the right-hand side are lower or equal than $1/2$, we obtain that $b'(t) - a'(t) \leq -(b(t) - a(t))$ and $b(t) - a(t) \to 0$ exponentially fast.

It follows that the distance between f_t and $\delta_{M_{(t)}}$ tends to 0, where $M_1(t) = \int df_t(p)p$ is the mean value of p. Assuming for the moment that

$$M_1'(t) = \frac{1}{2}(1 - 2M_1(t)), \tag{4.5}$$

it is clear that $M_1(t) \to 1/2$ and thus $f_t \to \delta_{\frac{1}{2}}$.

The proof (4.5) is easy. Taking $\varphi(p) = p$ in (4.3) gives

$$\begin{aligned} M_1'(t) = &\iint df_t(p)df_t(p')\mathbf{1}_{p'>p}\frac{1}{2p'}(-2pp' + 2p' - p) \\ &+ \iint df_t(p)df_t(p')\mathbf{1}_{p'<p}\frac{1}{2p}(-2p^2 + p') + \iint df_t(p)df_t(p')\mathbf{1}_{p'=p}\frac{1-2p}{2}. \end{aligned}$$

Exchanging p and p' in the second integral gives

$$\begin{aligned} M_1'(t) &= \iint df_t(p)df_t(p')\mathbf{1}_{p'>p}(1-p-p') + \iint df_t(p)df_t(p')\mathbf{1}_{p'=p}\frac{1-2p}{2} \\ &= \frac{1}{2}\iint df_t(p)df_t(p')(1-p-p'). \end{aligned}$$

Hence, the result follows.

4.3. *Second price auction*

For second price auction we take $c = 0$ and the field H_c takes the form

$$\begin{aligned} H_0[f_t](p) = &\int df_t(p')\mathbb{1}_{\{p'>p\}}\frac{1}{2p'}\left(p^2 - 2pp' + 2p' - p\right) \\ &+ \int df_t(p')\mathbb{1}_{\{p>p'\}}\frac{1}{2p}\left(-pp' + p'\right) \\ &+ \int df_t(p')\mathbb{1}_{\{p=p'\}}\frac{1-p}{2}. \end{aligned} \tag{4.6}$$

As before, we can show that the Nash equilibrium where all the bidders bid their true valuation is the only stationary solution in the functional form of a Dirac's Delta.

Theorem 4.2. *The unique Dirac's Delta solution of equation* (4.3) *for* $c = 0$ *is* $f = \delta_1$.

Proof. As before this follows noticing that $H_1[\delta_a](a) = \frac{1-a}{2}$ which is zero only when $a = 1$. □

In fact, following the same heuristic argument as before, we can convince ourselves that f_t converges to δ_1 as $t \to +\infty$. Indeed denoting $a(t)$ the left endpoint of the support of f_t, we have (informally) that

$$a'(t) = H_0[f_t](a(t)) = \int df_t(p')\mathbb{1}_{\{p'>p\}}\frac{1}{2p'}\left(a^2 - 2ap' + 2p' - a\right) + \frac{1-a}{2}f_t(\{a(t)\}$$

i.e.,

$$\begin{aligned} a'(t) &= (1-a(t))\Big\{\int df_t(p')\mathbb{1}_{\{p'>p\}}\frac{1}{2p'}(2p'-a) + \frac{1}{2}f_t(\{a(t)\}\Big) \\ &= (1-a(t))\int df_t(p')\frac{1}{2p'}(2p'-a) \end{aligned}$$

Since $(2p'-a)/(2p') \geq 1/2$ we obtain $a'(t) \geq \frac{1}{2}(1-a(t))$ so that $a(t) \to 1$ as desired.

4.4. *Simulations*

In this section we present numerical simulations of the discrete dynamics corresponding to the microscopic rules (4.1). We considered, as before, 1000 players and 1000 iterations, that is, 10^6 two-player game interactions. In both cases we choose an initial distribution drawn from a normal distribution $\mathcal{N}(0,1)$, considering only values in $[0,1]$. As before, we cannot expect a convergence to a pure Dirac's Delta function, due to random fluctuations and the step size.

We show the result for a first and second price auction in Figure 6 and Figure 7 respectively. We can observe that bidders' parameter p concentrate around 0.5 and 1 respectively, in agreement with the theoretical considerations exposed above.

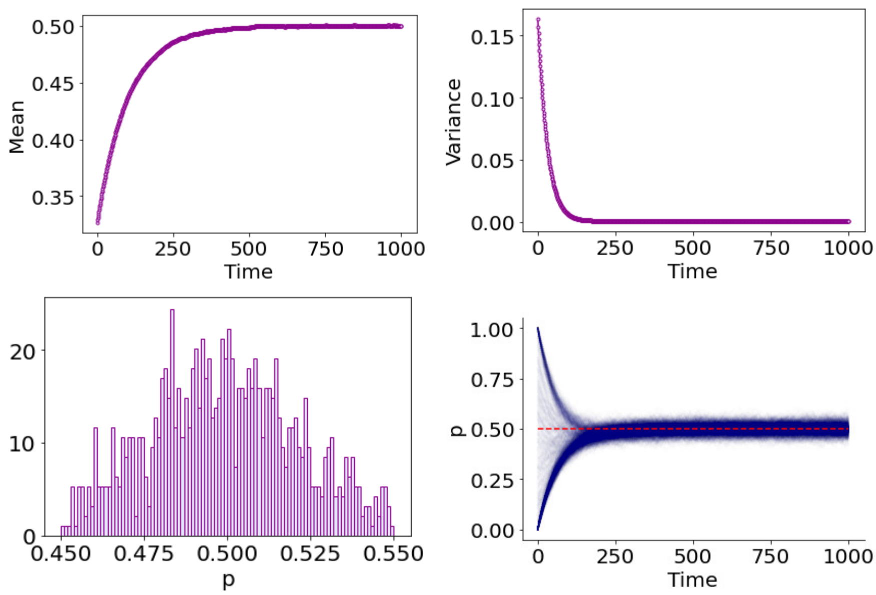

Fig. 6. Learning in the first price auction with rule (4.1). First row: evolution of the mean (left) and variance (right) of the distribution of the parameter p. Second row: histogram of the final density of players in the parameter space (left) and individual trajectories (right).

5. Final remarks

We have analyzed first and second price auctions within the kinetic theory of active particles. This powerful tool enables us to find Bolztmann-type equations for a population of agents interacting through auctions on the space of strategies, defined as the percentage of the true valuation that they will bid. In a simple model, using a posteriori the information of the true valuations, we proved the convergence of the dynamics to the Nash equlibrium. Then, by introducing a different

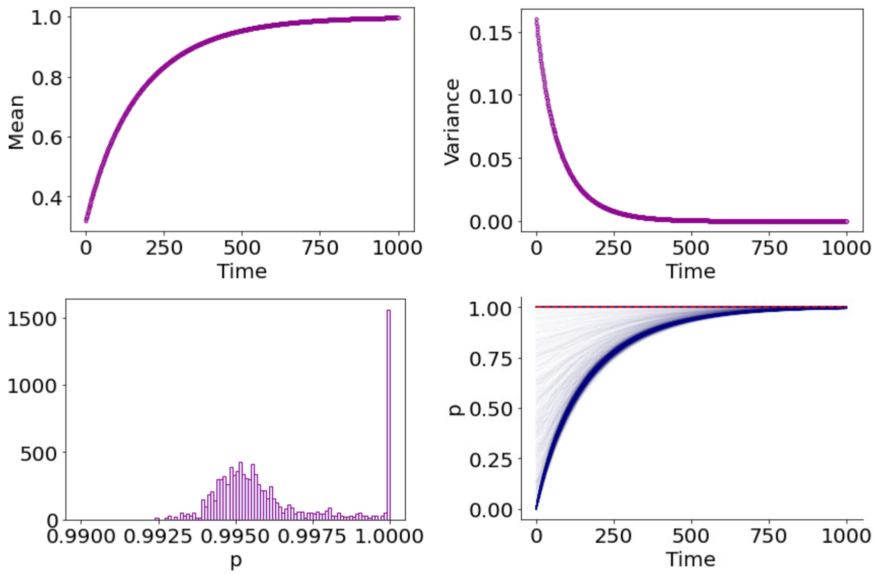

Fig. 7. Learning in the second price auction with rule (4.1). First row: evolution of the mean (left) and variance (right) of the distribution of the parameter p. Second row: histogram of the final density of players in the parameter space (left) and individual trajectories (right).

microscopic rule, we obtained that the Nash equilibria of both auctions are the support of Dirac delta functions which are the stationary solutions of the corresponding kinetic equations. We show the convergence by using agent based simulations.

The proposed model captures the essence of the kinetic theory of active particles (KTAP) by incorporating its key features. Bidders' interactions and resulting payoffs are inherently non-linear. This means small changes in strategies can have significant repercussions on outcomes, creating a dynamic and unpredictable environment. On the other hand, each bidder's behavior is unique, driven by their private valuation and past experiences. This diversity in bidding strategies fosters a complex interplay within the system. They continuously adapt their internal state (represented by their parameter) based on the outcomes of their interactions. This learning process allows them to refine their strategies and adjust to the evolving behavior of other bidders.

Furthermore, the model strengthens the connection between KTAP and game theory in two key ways:

- Game-mediated interactions: Bidders engage in repeated strategic interactions, akin to playing games, which drive the evolution of their internal activity (particles' activity, in KTAP terminology). This highlights the critical role of game theory in modeling strategic agent behavior.

- Dynamic learning towards Nash equilibrium: Through their learning process, bidders gravitate towards a dynamic Nash equilibrium, where no individual can benefit by unilaterally changing their strategy. This demonstrates how the model leverages KTAP's framework to capture the emergence of equilibrium outcomes in complex strategic systems.

There are several interesting remaining problems, let us mention a few of them:

5.1. *Three or more players*

Real world auctions typically have more than two bidders, and in that case the previous models can be analyzed through agent-based simulations.

We present here the results of some experiments varying the number of bidders. We introduce a microscopic update rule depending on the order of a player according to the value of its parameter p. Given the symmetry of players, we assume that a winner of the auction does not wish to belong to the upper half of players, and any player who loses, does not wish to belong to the lower half of players, and they decrease or increase their parameter p as before. If k is odd, the player in the middle of the list will flip a coin to decide which half of the list they belong to.

In Figure 8 we show the simulation for k players auctions, $2 \leq k \leq 15$ and using the initial condition $\mathcal{C}_1$. We take the mean value of p after 800 steps of the dynamics, and compare with the theoretical Nash equilibrium. We omit the histograms for each k, although we can observe that the population concentrates around the corresponding Nash equilibrium for each k.

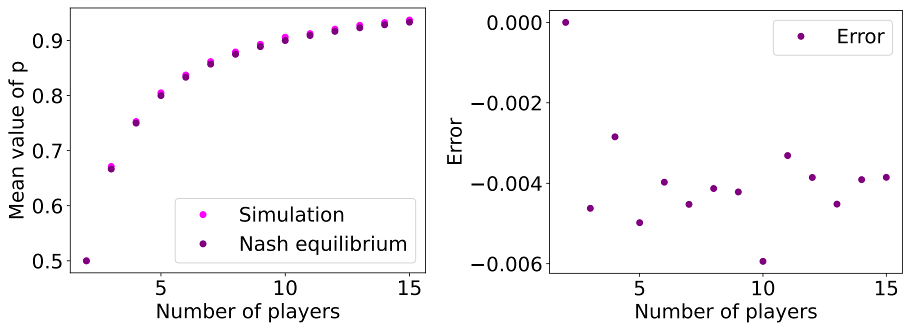

Fig. 8. Simulation of k players auctions. Nash equilibrium and mean value of p after 800 steps of the evolution with $1000 players$, $q = 0.9$, and $2 \leq k \leq 15$ (left). Difference between the theoretical Nash equilibrium and the mean value (right).

Finally, let us remark that the derivation of the corresponding kinetic equations involves a highly complex system of interactions, since the interaction rules depend on the parameters of the different bidders.

5.2. *Other auctions formats*

There are several auction formats beyond the ones considered here. A very interesting problem is how to model in agent-based frameworks, an English auction. Even for two bidders A and B, the dynamics of the interaction is not trivial. Suppose that each bidder has a preliminary offer, and after agent A places a bid, agent B can offer a higher number than its preliminary offer, which in turns can be followed by a higher bid of agent A, and so on. In this way, a single interaction is defined by a sequence of interactions where the previous information is incorporated in the new bids.

For instance, few works deal with iterative auctions using an agent-based approach, and we believe it is very important to understand this dynamics. Essentially, two time scales seems to be involved, a large one where the active particles learn how to bid, and a shorter one where they change their internal mechanisms quickly reacting to the other bids.

An interesting format is the *all pay auction*,[25] where all the bidders pay while the object is only assigned to the one with the highest bid. In that case, the optimal strategy for symmetric bidders with independent and identical random uniform valuations is to bid $1/N$ of the true valuation of the object, where N is the number of bidders.

5.3. *Non-homogeneous bidders*

We have considered only symmetric auctions with the same class of bidders, i.e., risk neutral and drawing their valuations from the same distribution.

However, there are many types of bidders, which can be classified in terms of their risk attitude, distribution of valuations, and many other factors. For instance, in procurement auctions, where bid compete for some contract, bidder's financial size, credit access or budget, previous experience, among others, make the problem difficult to solve even from the theoretical point of view.

5.4. *Theoretical considerations*

In a recent work,[33] a learning rule for renewable energy procurement auctions was considered. As in the present work, a rigorous derivation of the Fokker-Planck equations seems difficult to achieve since the terms in the Boltzmann-type equation are not continuous. In this class of auctions, each player bids for a portion or the total of the auctioned object, and one or more bidders can be awarded. Although simulations in that work reproduce the results of German wind and solar PV auctions with notable accuracy, a complete theoretical analysis and understanding of these auctions has not yet been fully developed.

Acknowledgment

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